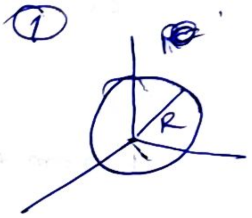
Volume and Surface area of an n-dimensional sphere of radius R. —

Consider an n-dimensional space in which the position of a point is denoted by the vector $\vec{r}$ with Cartesian components $(x_1, ---, x_n)$. The volume element $d\Omega_n$ in this space would be

$$d^n\vec{r} = d\Omega_n = dx_1\, dx_2 \cdots dx_n \qquad \text{—— (1)}$$



Volume will be given by

$$\Omega_n(R) = \underset{x_1^2+x_2^2+\cdots+x_n^2 \le R^2}{\int dx_1 \cdots \int dx_n} = C_n R^n \qquad \text{—— (2)}$$

Since Ω_n ~~Since~~ will be proportional to R^n

C_n = const. → depends on dimensionality of space.

From (2)

$$d\Omega_n(R) = S_n(R)\, dR \quad , \quad S_n \rightarrow \text{surface area}$$

or $$d\Omega_n(R) = n C_n R^{n-1}\, dR \qquad \text{—— (3)}$$

Since we have the known formula for the integral

$$\int_{-\infty}^{+\infty} dx_1 \cdots \int_{-\infty}^{+\infty} dx_n\, e^{-(x_1^2+\cdots+x_n^2)} = \left[\int_{-\infty}^{+\infty} dx\, e^{-x^2}\right]^n$$

$$= \pi^{n/2} \qquad \text{—— (4)}$$

Rewriting eq. (4) in the form

$$\pi^{n/2} = \int_{x_i=-\infty}^{+\infty} dx_1 \cdots \int \exp\left[-\sum_{i=1}^{n} x_i^2\right] dx_n$$

~~or $\pi^{n/2}$~~ In spherical Coordinates

$$\pi^{n/2} = \int_0^{\infty} dR \; n\, C_n R^{n-1} e^{-R^2}$$

$$= n\, C_n \frac{1}{2} \Gamma\left(\frac{n}{2}\right)$$

$$= C_n \cdot \Gamma\left(\frac{n}{2}+1\right)$$

or $$\boxed{C_n = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)}} \quad \text{— (5)}$$

$\Gamma \rightarrow$ denotes here the gamma function

~~$\Gamma(n+1) = n(n-1) \cdots \Gamma(n)$~~

$\Gamma(\nu) = \frac{1}{\nu}\Gamma(\nu+1)$, $0 < \nu \leq 1$

where we have used the relation

$$\int_0^{\infty} e^{-\alpha y^2} y^{\nu} dy = \frac{1}{2\alpha^{(\nu+1)/2}} \Gamma\left(\frac{\nu+1}{2}\right), \quad \nu > -1$$

for $\alpha = 1$ and $\nu = n-1$

$$\boxed{\int_0^{\infty} e^{-y^2} y^{n-1} dy = \frac{1}{2}\Gamma(n)}$$

~~Thus $\Omega_n(R) =$~~

We can write (5) as $$C_n = \frac{\pi^{n/2}}{\left(\frac{n}{2}\right)!}$$

Thus $$\boxed{\Omega_n(R) = \frac{\pi^{n/2}}{\left(\frac{n}{2}\right)!} R^n}$$ and $$\boxed{S_n(R) = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)} R^{n-1}}$$